New Analytical Solutions to the Nonlinear Schrödinger Equation Model

Yuanyuan Zhang, Ying Zheng, and Hongqing Zhang

Department of Applied Mathematics, Dalian University of Technology, Dalian, 116024, P. R. China

Reprint requests to Yu. Z.; E-mail: mathzhyy@yahoo.com.cn

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In this paper, new analytical solutions of the nonlinear Schrödinger equation model are obtained. The properties of the new exact solutions are shown by some figures.

Key words: Analytical Solutions; Nonlinear Schrödinger Equation; Generalized Riccati Equation.

1. Introduction

In recent years, searching for exact analytical solutions to nonlinear evolution equations (NEEs) has attracted considerable attention in mathematics and physics. Exact solutions, which are explicit, may help physicists and engineers to discuss and examine the sensitivity of a model with respect to physical parameters. With the development of soliton theory, many powerful methods have been proposed to find solitary wave solutions to NEEs, such as the inverse scattering method [1, 2], the truncated Painlevé expansion method [3], the Hirota bilinear method [4], the tanh method [5 – 8], and various generalized Riccati equation expansion methods [9 – 12].

The nonlinear Schrödinger equation model (NLSE) is one of the most important and universal nonlinear models of modern science. Since solitary waves, or solitons, which are the best known solutions of NLSE, were introduced and developed in 1971 by Zakharov and Shabat [13], there have been many significant contributions to the development of the NLSE solitons theory [13 – 30]. In optical communication systems, the transmission of solitons is described by the NLSE model with space-dependent coefficients:

$$\mathrm{i}\frac{\partial u}{\partial z}+\frac{1}{2}\beta(z)\frac{\partial^2 u}{\partial t^2}+\delta(z)|u|^2u=\mathrm{i}\alpha(z)u, \qquad (1)$$

where $\beta(z)$ and $\delta(z)$ are the slowly increasing dispersion and nonlinearity coefficient, respectively; $\alpha(z)$ represents the heat-insulating amplification or loss. Serkin and Hasegawa [21, 22] developed an effective mathematical algorithm to discover and investigate an infinite number of novel soliton solutions of (1) and discussed the problem of soliton management described by (1). Ruan and Chen [27] studied (1) by a symmetry approach and reported some exact solutions of it. Recently, Li [31, 32] proposed a new algebraic method to study an averaged dispersion-managed (DM) fiber system equation and the NLSE (1), and obtained rich new exact solutions of them.

In this paper, we improve the generalized Riccati equation rational expansion method in [12] to solve the NLSE (1). Further the properties of the new exact solutions are shown by some figures. We show that the solutions we get are more general than those yielded by the generalized Riccati equation rational expansion method, and may enable one to better understand the physical phenomena which (1) describes. In fact, our method is also powerful to solve other nonlinear evolution equations.

The paper is organized as follows. In Section 2 a method for constructing analytical solutions of (1) is established. In Section 3 a rich variety of analytical solutions of (1) is obtained, and the main features of these solutions are investigated by using computer simulation. Section 4 contains a short summary and discussion.

2. Summary of our Method

The main steps of our method are:

Step 1. Given a nonlinear NEE with the variables z and t:

$$p(u_t,u_z,u_{zt},u_{tt},u_{zz}\cdots)=0, \qquad (2)$$

we assume that the solution of (2) is as follows:

$$u(z,t)=a_0+\sum_{i=1}^{m}\frac{a_i\phi^i(\xi)+b_i\phi^{i-1}(\xi)\phi'(\xi)}{(1+\mu_1\phi(\xi)+\mu_2\phi'(\xi))^i}, \qquad (3)$$

where the parameter m can be determined by balancing the highest order derivative term and the nonlinear terms in (2); $a_0 = a_0(z,t)$, $a_i = a_i(z,t)$, $b_i = b_i(z,t)$ $(i = 1,\cdots,m)$, $\xi = \xi(z,t)$ are all differentiable functions of z,t; $\phi(\xi)$ and $\phi'(\xi)$ satisfy the generalized Riccati equation (4) as follows:

$$\phi' - (h_1 + h_2\phi^2) = \frac{\mathrm{d}\phi}{\mathrm{d}\xi} - (h_1 + h_2\phi^2) = 0. \quad (4)$$

We know that (4) has the following solutions:

(a) When $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$,

$$\begin{aligned}\phi(\xi) &= \tanh(\xi) \pm \mathrm{i}\,\mathrm{sech}(\xi),\\ \phi(\xi) &= \coth(\xi) \pm \mathrm{csch}(\xi).\end{aligned} \quad (5)$$

(b) When $h_1 = h_2 = \pm\frac{1}{2}$,

$$\begin{aligned}\phi(\xi) &= \sec(\xi) \pm \tan(\xi),\\ \phi(\xi) &= \csc(\xi) \pm \cot(\xi).\end{aligned} \quad (6)$$

(c) When $h_1 = 1$ and $h_2 = -1$,

$$\phi(\xi) = \tanh(\xi), \quad \phi(\xi) = \coth(\xi). \quad (7)$$

(d) When $h_1 = h_2 = 1$,

$$\phi(\xi) = \tan(\xi). \quad (8)$$

(e) When $h_1 = h_2 = -1$,

$$\phi(\xi) = \cot(\xi). \quad (9)$$

(f) When $h_1 = 0$ and $h_2 \neq 0$,

$$\phi(\xi) = -\frac{1}{h_2\xi + C_0}, \quad (10)$$

where $C_0 = \text{const.}$

Step 2. Substituting (3) along with (4) into (2), we obtain a set of algebraic polynomials for $\phi^i(\xi)$ $(i = 0,1,\cdots,)$. Setting the coefficients of these terms $\phi^i(\xi)$ to zero, we get a system of over-determined partial differential equations (PDEs) or ordinary differential equations (ODEs) with respect to the unknown functions a_0, a_i, b_i $(i = 1,\cdots,m)$, ξ.

Step 3. Solving the over-determined PDE (or ODE) system by use of a symbolic computation system (like Maple), we would end up with explicit expressions for μ_1, μ_2, a_0, a_i, b_i $(i = 1,2,\cdots,m)$ and ξ or the constraints among them.

Step 4. Thus according to (3), (4) and the conclusion in Step 3, we can obtain many families of analytical solutions for (2).

Remark 1. Through the description above, we can find that our method is more practical and convenient than the method in [12], because we suppose that the coefficients of the ansatz (3) are not undetermined constants but undetermined functions. This can lead to more general solutions of the NEEs.

3. Analytical Solutions and Computer Simulations

We now investigate the NLSE (1) with our algorithm. In order to obtain some exact solutions of the NLSE (1), we first make the transformation

$$u(z,t) = V(z,t)\exp[\mathrm{i}\Theta(z,t)]. \quad (11)$$

Then, substituting (11) into (1) and setting the real and imaginary parts of the resulting equation equal to zero, we obtain the following sets of PDEs:

$$-V\Theta_z + \frac{1}{2}\beta(z)(V_{tt} - V\Theta_t^2) + \delta(z)V^3 = 0, \quad (12)$$

$$V_z + \frac{1}{2}\beta(z)(2V_t\Theta_t + V\Theta_{tt}) - \alpha(z)V = 0. \quad (13)$$

By balancing V_{tt} and V^3 in (12), we obtain $m = 1$. Therefore we try to solve (12) and (13) in the following special form:

$$u(z,t) = a_0(z) + \frac{a_1(z)\phi(\xi) + b_1(z)\phi(\xi)\phi'(\xi)}{1 + \mu_1\phi(\xi) + \mu_2\phi'(\xi)}, \quad (14)$$

$$\Theta(z,t) = t^2\Delta(z) + t\Gamma(z) + \Omega(z), \quad (15)$$

$$\xi = t\lambda(z) + \eta(z), \quad (16)$$

where $a_0(z)$, $a_1(z)$, $b_1(z)$, $\Delta(z)$, $\Gamma(z)$, $\Omega(z)$, $\lambda(z)$ and $\eta(z)$ are functions to be determined, and $\phi(\xi)$ and $\phi'(\xi)$ satisfy the generalized Riccati equation (4).

Remark 2. For a simple and convenient computation, we have set $a_0(z,t) = a_0(z)$, $a_1(z,t) = a_1(z)$, $b_1(z,t) = b_1(z)$, $\xi(z,t) = t\lambda(z) + \eta(z)$.

Substituting (4), (14), (15), and (16) into (12) and (13), collecting the coefficients of the polynomials of $\phi(\xi)$ and t of the resulting system, then setting each coefficient to zero, we obtain an over-determined ODE system with respect to the differentiable functions $\{a_0(z), a_1(z), b_1(z), \Delta(z), \Gamma(z), \Omega(z), \lambda(z), \eta(z), \alpha(z), \beta(z), \delta(z)\}$. For simplicity we do not list them

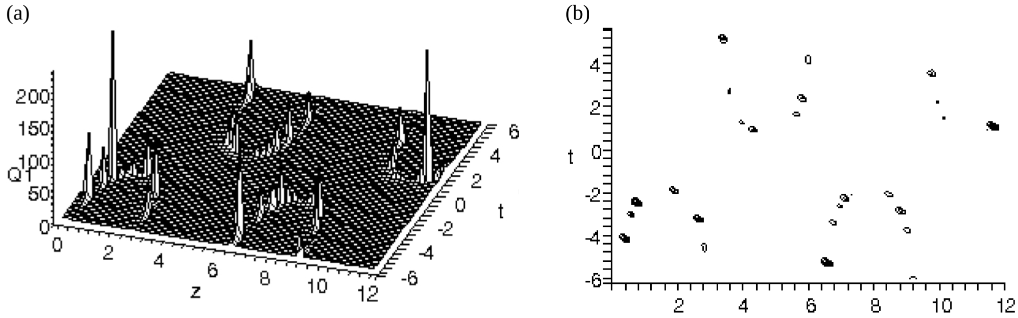

Fig. 1. Exact solution u_{12} [i. e. (18)], where the parameters are as follows: $\Delta(z)=\sin(z)$, $b_1(z)=\cos(z)$, $\mu_2=1$, $C_1=0$, $C_2=0$, $C_3=1$, $C_4=1$.

here. Solving the ODE system with Maple, we obtain the following results.

Case 1

$$\beta(z)=-\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)}{(\Delta(z))^2},$$
$$a_1(z)=\pm\frac{\sqrt{-\mu_2(\mu_2h_1+1)h_2}\,b_1(z)}{\mu_2^2h_1+\mu_2},$$

$$\Omega(z)=\int-\frac{1}{4}\frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)\left(2h_2(\lambda(z))^2h_1-(\Gamma(z))^2\right)}{(\Delta(z))^2}\mathrm{d}z+C_2,$$

$$\mu_1=0,\quad a_0(z)=0,$$
$$\delta(z)=-\frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)\lambda^2(z)(h_1h_2\mu_2^2+h_2\mu_2)}{(\Delta(z))^2(b_1(z))^2},$$
$$\eta(z)=\int\frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)\lambda(z)\Gamma(z)}{(\Delta(z))^2}\mathrm{d}z+C_1,$$
$$\lambda(z)=C_3\Delta(z),$$
$$\alpha(z)=\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}z}b_1(z)\right)\Delta(z)-\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)b_1(z)}{2\Delta(z)b_1(z)},$$
$$\Gamma(z)=C_4\Delta(z),$$

where $\Delta(z)$ and $b_1(z)$ are arbitrary functions of z, and C_1, C_2, C_3, C_4, μ_2 are arbitrary constants.

Case 2

$$\alpha(z)=\frac{\frac{\mathrm{d}}{\mathrm{d}z}b_1(z)}{b_1(z)},\quad \Delta(z)=0,$$

$$a_1(z)=\pm\frac{\sqrt{-\mu_2(\mu_2h_1+1)h_2}\,b_1(z)}{\mu_2^2h_1+\mu_2},$$

$$\eta(z)=\int-\beta(z)\lambda(z)\Gamma(z)\mathrm{d}z+C_1,$$
$$\Gamma(z)=C_4,\quad \lambda(z)=C_3,$$
$$\Omega(z)=\int\left(h_2(\lambda(z))^2h_1-\frac{1}{2}(\Gamma(z))^2\right)\beta(z)\mathrm{d}z+C_2,$$
$$a_0(z)=0,$$
$$\mu_1=0,\quad \delta(z)=\frac{\beta(z)\lambda^2(z)(\mu_2^2h_2h_1+h_2\mu_2)}{(b_1(z))^2},$$

where $\beta(z)$ and $b_1(z)$ are arbitrary functions of z, and C_1, C_2, C_3, C_4, μ_2 are arbitrary constants.

Case 3

$$a_1(z)=0,\quad \beta(z)=-\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)}{(\Delta(z))^2},$$
$$\lambda(z)=C_4\Delta(z),\quad \mu_2=-\frac{1}{2h_1},$$

$$\Omega(z)=\int\frac{\left(h_2(\lambda(z))^2h_1+\frac{1}{4}(\Gamma(z))^2\right)\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)}{(\Delta(z))^2}\mathrm{d}z+C_2,$$

$$a_0(z)=0,\quad \delta(z)=\frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)(\lambda(z))^2(h_2+\mu_1^2h_1)}{(\Delta(z))^2(b_1(z))^2h_1},$$

$$\eta(z)=\int\frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)\lambda(z)\Gamma(z)}{(\Delta(z))^2}\mathrm{d}z+C_1,$$
$$\Gamma(z)=C_3\Delta(z),$$
$$\alpha(z)=\frac{1}{2}\frac{2\left(\frac{\mathrm{d}}{\mathrm{d}z}b_1(z)\right)\Delta(z)-\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)b_1(z)}{\Delta(z)b_1(z)},$$

where $\Delta(z)$ and $b_1(z)$ are arbitrary functions of z, and C_1, C_2, C_3, C_4, μ_1 are arbitrary constants.

With these results we get the general form (11) of solutions of (1).

Family 1. According to Case 1, when $h_1 = \frac{1}{2}$ and $h_2 = -\frac{1}{2}$ we obtain the solutions for the NLSE as follows (Fig. 1):

$$u_{11}(z,t) = \frac{\pm\frac{b_1(z)}{\sqrt{\mu_2^2+2\mu_2}}(\tanh(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi)) + b_1(z)(\mathrm{sech}^2(\xi)\mp \mathrm{i}\,\mathrm{sech}(\xi)\tanh(\xi))}{1+\mu_2(\mathrm{sech}^2(\xi)\pm \mathrm{i}\,\mathrm{sech}(\xi)\tanh(\xi))} \times \exp\left\{\mathrm{i}(\Delta(z)t^2 + C_4\Delta(z)t + \int \frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)(\lambda^2(z)+2\Gamma^2(z))}{8\Delta^2(z)}\mathrm{dz} + C_2)\right\}, \quad (17)$$

$$u_{12}(z,t) = \frac{\pm\frac{b_1(z)}{\sqrt{\mu_2^2+2\mu_2}}(\coth(\xi)\pm \mathrm{csch}(\xi)) - b_1(z)(\mathrm{csch}^2(\xi)\pm \mathrm{csch}(\xi)\coth(\xi))}{1-\mu_2(\mathrm{csch}^2\xi \pm \mathrm{csch}(\xi)\coth(\xi))} \times \exp\left\{\mathrm{i}\left(\Delta(z)t^2 + C_4\Delta(z)t + \int \frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)(\lambda^2(z)+2\Gamma^2(z))}{8\Delta^2(z)}\mathrm{dz} + C_2\right)\right\}, \quad (18)$$

where $\mu_2, C_1, C_2, C_3, C_4$ are arbitrary constants, $\Delta(z)$, $b_1(z)$ are arbitrary functions,

$$\xi = C_3\Delta(z)t + \int \frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)\lambda(z)\Gamma(z)}{(\Delta(z))^2}\mathrm{dz} + C_1, \quad \beta(z) = -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)}{(\Delta(z))^2},$$

$$\delta(z) = \frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)\lambda^2(z)(\mu_2^2+2\mu_2)}{8(\Delta(z))^2(b_1(z))^2}, \quad \alpha(z) = \frac{2\left(\frac{\mathrm{d}}{\mathrm{dz}}b_1(z)\right)\Delta(z) - \left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)b_1(z)}{2\Delta(z)b_1(z)}.$$

Family 2. According to Case 1, when $h_1 = h_2 = \pm\frac{1}{2}$, we obtain the solutions for the NLSE as follows:

$$u_{21}(z,t) = \frac{\mp\frac{b_1(z)}{\sqrt{\mp\mu_2^2-2\mu_2}}(\sec(\xi)\pm\tan(\xi)) + b_1(z)(\sec(\xi)\tan(\xi)\pm\sec^2(\xi))}{1+\mu_2(\sec(\xi)\tan(\xi)\pm\sec^2(\xi))} \times \exp\left\{\mathrm{i}(\Delta(z)t^2 + C_4\Delta(z)t + \int \frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)(\mp\lambda^2(z)+2\Gamma^2(z))}{8\Delta^2(z)}\mathrm{dz} + C_2)\right\}, \quad (19)$$

$$u_{22}(z,t) = \frac{\mp\frac{b_1(z)}{\sqrt{\mp\mu_2^2-2\mu_2}}(\csc(\xi)\pm\cot(\xi)) - b_1(z)(\csc(\xi)\cot(\xi)\pm\csc^2(\xi))}{1-\mu_2(\csc(\xi)\cot(\xi)\pm\csc^2(\xi))} \times \exp\left\{\mathrm{i}\left(\Delta(z)t^2 + C_4\Delta(z)t + \int \frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)(\mp\lambda^2(z)+2\Gamma^2(z))}{8\Delta^2(z)}\mathrm{dz} + C_2\right)\right\}, \quad (20)$$

where $\mu_2, C_1, C_2, C_3, C_4$ are arbitrary constants, $\Delta(z)$, $b_1(z)$ are arbitrary functions,

$$\xi = C_3\Delta(z)t + \int \frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)\lambda(z)\Gamma(z)}{(\Delta(z))^2}\mathrm{dz} + C_1, \quad \beta(z) = -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)}{(\Delta(z))^2},$$

$$\delta(z) = -\frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)\lambda^2(z)(\mu_2^2\pm 2\mu_2)}{8(\Delta(z))^2(b_1(z))^2}, \quad \alpha(z) = \frac{2\left(\frac{\mathrm{d}}{\mathrm{dz}}b_1(z)\right)\Delta(z) - \left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)b_1(z)}{2\Delta(z)b_1(z)}.$$

Family 3. According to Case 1, when $h_1 = 1$ and $h_2 = -1$, we obtain the solutions for the NLSE as follows (Fig. 2):

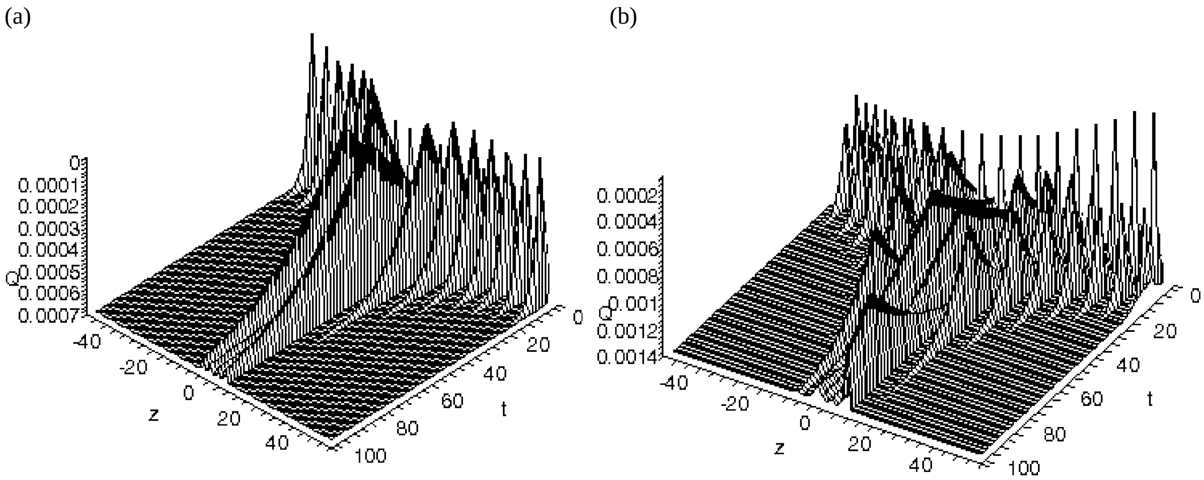

Fig. 2. (a) Propagation scenario of the quasi-solitons given by u_{31} [i. e. (21)] after choosing $\Delta(z) = \cos(z)$, $b_1(z) = 0.001$, $\mu_2 = 1$, $C_1 = 0$, $C_2 = 0$, $C_3 = 1$, $C_4 = 1$; (b) interaction of these quasi-solitons with different periodically varying control functions $\Delta = \cos(z)$ and $\Delta' = \sin(z+\frac{\pi}{4})$.

$$u_{31}(z,t) = \frac{\pm\frac{b_1(z)}{\sqrt{\mu_2^2+\mu_2}}\tanh(\xi) + b_1(z)\,\mathrm{sech}^2(\xi)}{1+\mu_2\,\mathrm{sech}^2(\xi)} \times \exp\left\{ \mathrm{i}\left(\Delta(z)t^2 + C_4\Delta(z)t + \int \frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)(2\lambda^2(z)+\Gamma^2(z))}{4\Delta^2(z)}\mathrm{d}z + C_2\right)\right\}, \tag{21}$$

$$u_{32}(z,t) = \frac{\pm\frac{b_1(z)}{\sqrt{\mu_2^2+\mu_2}}\coth(\xi) - b_1(z)\,\mathrm{csch}^2(\xi)}{1-\mu_2\,\mathrm{csch}^2(\xi)} \times \exp\left\{ \mathrm{i}\left(\Delta(z)t^2 + C_4\Delta(z)t + \int \frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)(2\lambda^2(z)+\Gamma^2(z))}{4\Delta^2(z)}\mathrm{d}z + C_2\right)\right\}, \tag{22}$$

where μ_2, C_1, C_2, C_3, C_4 are arbitrary constants, $\Delta(z)$, $b_1(z)$ are arbitrary functions,

$$\xi = C_3\Delta(z)t + \int \frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)\lambda(z)\Gamma(z)}{(\Delta(z))^2}\mathrm{d}z + C_1, \quad \beta(z) = -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)}{(\Delta(z))^2},$$

$$\delta(z) = \frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)\lambda^2(z)(\mu_2^2+\mu_2)}{2(\Delta(z))^2(b_1(z))^2}, \quad \alpha(z) = \frac{2\left(\frac{\mathrm{d}}{\mathrm{d}z}b_1(z)\right)\Delta(z) - \left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)b_1(z)}{2\Delta(z)b_1(z)}.$$

Family 4. According to Case 1, when $h_1 = h_2 = 1$, we obtain the solutions for the NLSE as follows (Fig. 3):

$$u_4(z,t) = \frac{\mp\frac{b_1(z)}{\sqrt{-\mu_2^2-\mu_2}}\tan(\xi) + b_1(z)\sec^2(\xi)}{1+\mu_2\sec^2(\xi)} \times \exp\left\{ \mathrm{i}\left(\Delta(z)t^2 + C_4\Delta(z)t + \int \frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)(-2\lambda^2(z)+\Gamma^2(z))}{4\Delta^2(z)}\mathrm{d}z + C_2\right)\right\}, \tag{23}$$

where μ_2, C_1, C_2, C_3, C_4 are arbitrary constants, $\Delta(z)$, $b_1(z)$ are arbitrary functions,

$$\xi = C_3\Delta(z)t + \int \frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)\right)\lambda(z)\Gamma(z)}{(\Delta(z))^2}\mathrm{d}z + C_1, \quad \beta(z) = -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{d}z}\Delta(z)}{(\Delta(z))^2},$$

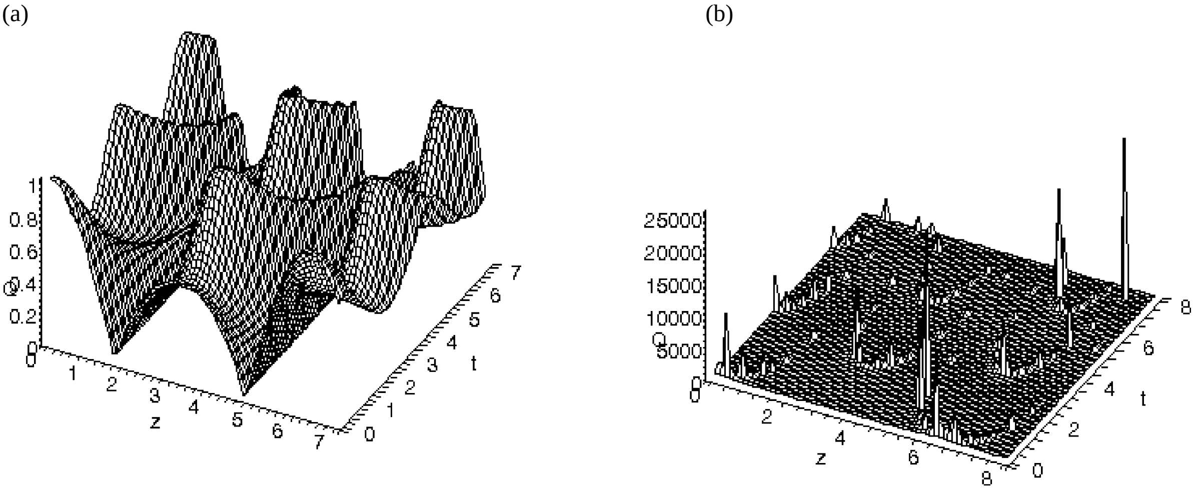

Fig. 3. Exact solution u_4 [i. e. (23)]; (a) $\Delta(z) = \cos(z)$, $b_1(z) = \cos(z)$, $\mu_2 = 1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 1$, $C_4 = 1$; (b) $\Delta(z) = \cos(z)$, $b_1(z) = -\cos(z)$, $\mu_2 = -0.001$, $C_1 = 1$, $C_2 = 2$, $C_3 = 1$, $C_4 = 1$.

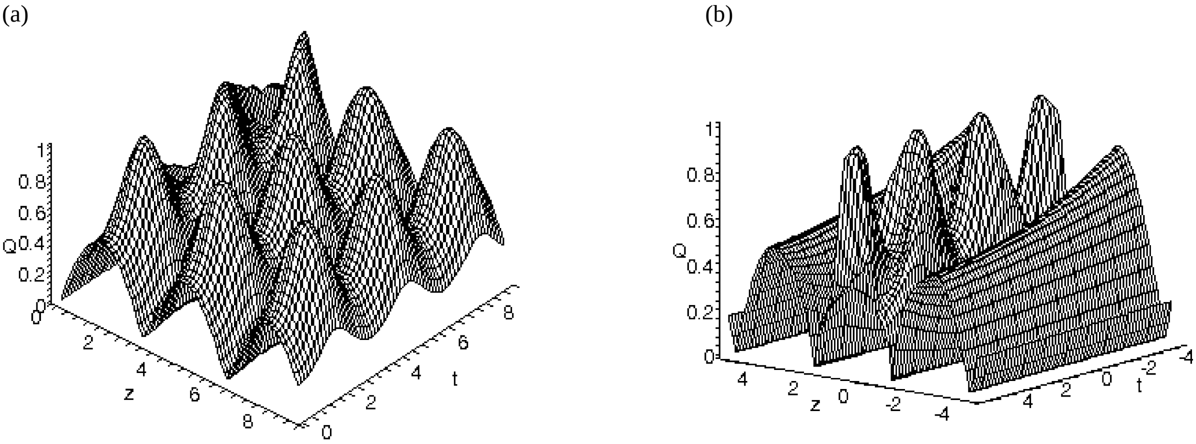

Fig. 4. Exact solution u_5 [i. e. (24)]; (a) $\Delta(z) = \tanh(z)$, $b_1(z) = \sin(z)$, $\mu_2 = -1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 1$, $C_4 = 1$; (b) $\Delta(z) = \operatorname{sech}(z)$, $b_1(z) = \cos(z)$, $\mu_2 = -1$, $C_1 = 1$, $C_2 = 2$, $C_3 = 1$, $C_4 = 1$.

$$\delta(z) = -\frac{\left(\frac{d}{dz}\Delta(z)\right)\lambda^2(z)(\mu_2^2+\mu_2)}{2(\Delta(z))^2(b_1(z))^2}, \quad \alpha(z) = \frac{2\left(\frac{d}{dz}b_1(z)\right)\Delta(z) - \left(\frac{d}{dz}\Delta(z)\right)b_1(z)}{2\Delta(z)b_1(z)}.$$

Family 5. According to Case 1, when $h_1 = h_2 = -1$, we obtain the solutions for the NLSE as follows (Fig. 4):

$$u_5(z,t) = \frac{\pm\frac{b_1(z)}{\sqrt{-\mu_2^2+\mu_2}}\cot(\xi) - b_1(z)\csc^2(\xi)}{1-\mu_2\csc^2(\xi)} \times \exp\left\{ i(\Delta(z)t^2 + C_4\Delta(z)t + \int \frac{\left(\frac{d}{dz}\Delta(z)\right)(-2\lambda^2(z)+\Gamma^2(z))}{4\Delta^2(z)}dz + C_2) \right\}, \tag{24}$$

where $\mu_2, C_1, C_2, C_3, C_4$ are arbitrary constants, $\Delta(z)$, $b_1(z)$ are arbitrary functions,

$$\xi = C_3\Delta(z)t + \int \frac{1}{2}\frac{\left(\frac{d}{dz}\Delta(z)\right)\lambda(z)\Gamma(z)}{(\Delta(z))^2}dz + C_1, \quad \beta(z) = -\frac{1}{2}\frac{\frac{d}{dz}\Delta(z)}{(\Delta(z))^2},$$

$$\delta(z) = -\frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)\lambda^2(z)(\mu_2^2-\mu_2)}{2(\Delta(z))^2(b_1(z))^2}, \quad \alpha(z) = \frac{2\left(\frac{\mathrm{d}}{\mathrm{dz}}b_1(z)\right)\Delta(z)-\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)b_1(z)}{2\Delta(z)b_1(z)}.$$

Family 6. According to Case 1, when $h_1 = 0$ and $h_2 \neq 0$, we obtain the solutions for the NLSE as follows:

$$u_6(z,t) = \frac{\frac{\pm\sqrt{-\mu_2 h_2}b_1(z)}{\mu_2}(h_2\xi+C_0)-b_1(z)h_2}{(h_2\xi+C_0)-\mu_2 h_2} \times \exp\left\{\mathrm{i}\left(\Delta(z)t^2+C_4\Delta(z)t+\int\frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)\Gamma^2(z)}{4\Delta^2(z)}\mathrm{dz}+C_2\right)\right\}, \tag{25}$$

where μ_2, C_1, C_2, C_3, C_4 are arbitrary constants, $\Delta(z)$, $b_1(z)$ are arbitrary functions,

$$\xi = C_3\Delta(z)t+\int\frac{1}{2}\frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)\lambda(z)\Gamma(z)}{(\Delta(z))^2}\mathrm{dz}+C_1, \quad \beta(z) = -\frac{1}{2}\frac{\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)}{(\Delta(z))^2},$$

$$\delta(z) = -\frac{\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)\lambda^2(z)(h_2\mu_2)}{2(\Delta(z))^2(b_1(z))^2}, \quad \alpha(z) = \frac{2\left(\frac{\mathrm{d}}{\mathrm{dz}}b_1(z)\right)\Delta(z)-\left(\frac{\mathrm{d}}{\mathrm{dz}}\Delta(z)\right)b_1(z)}{2\Delta(z)b_1(z)}.$$

Remark 3. We can also derive some other analytical solutions if we make use of the other two cases.

Remark 4. It is necessary to point out that our ansatz differs from the method proposed by Li [32], and we can obtain new exact solutions which can not be gotten by Li's method.

We have used direct computer simulation to investigate the main features of the analytical solutions obtained in our paper.

4. Conclusion and Discussion

In summary, using symbolic computation we have improved the generalized Riccati equation rational expansion method in [12] and proposed an improved method to solve the nonlinear Schrödinger equation model (NLSE). More importantly, we obtained several new analytical solutions. Furthermore, the properties of the new exact solutions are shown by some figures. We hope, the method given here is useful to study the soliton phenomena. The method can be used to solve many other nonlinear evolution equations or coupled ones.

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